A COUNTEREXAMPLE TO THE MIZOHATA-TAKEUCHI CONJECTURE

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ABSTRACT. We derive a family of L^p estimates of the X-Ray transform of positive measures in $\mathbb{R}^d$, which we use to construct a $\log R$-loss counterexample to the Mizohata-Takeuchi conjecture for every C^2 hypersurface in $\mathbb{R}^d$ that does not lie in a hyperplane. In particular, multilinear restriction estimates at the endpoint cannot be sharpened directly by the Mizohata-Takeuchi conjecture.

1. INTRODUCTION

Let $\Sigma \subset \mathbb{R}^d$ be a compact C^2 hypersurface with measure $\mathrm{d}s$. The *extension* operator $\mathcal{E} : L^1(\Sigma; \mathrm{d}s) \to L^\infty(\mathbb{R}^d)$ is defined by

$$[\mathcal{E}f](x) := \int_\Sigma e^{-2\pi i \langle x, \varsigma \rangle} f(\varsigma) \mathrm{d}\sigma(\varsigma)$$

The Mizohata-Takeuchi conjecture can be stated as follows:

Conjecture 1.1 (Mizohata-Takeuchi). Let Σ be any C^2 hypersurface in $\mathbb{R}^d$ with surface measure $\mathrm{d}\sigma$. Let $f \in L^2(\Sigma, \mathrm{d}\sigma)$ and let $w : \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ be a nonnegative weight. Then we have

$$\int_{\mathbb{R}^d} |\mathcal{E}f(x)|^2 w(x) \mathrm{d}x \lesssim \|f\|_{L^2(\Sigma; \mathrm{d}\sigma)}^2 \|Xw\|_{L^\infty} \tag{1.1}$$

where Xw denotes the X-Ray transform of w.

The primary result of this paper is the following counterexample to the Mizohata-Takeuchi conjecture with $\log R$-loss:

Theorem 1.2 (Counterexample). For any C^2 hypersurface Σ that is not a plane, there is some $f \in L^2(\Sigma; \mathrm{d}\sigma)$ and nonnegative weight $w_R \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ so that the following holds.

$$\int_{B_R(0)} |\mathcal{E}f(x)|^2 w(x) \mathrm{d}x \gtrsim \log R \|f\|_{L^2(\Sigma; \mathrm{d}\sigma)}^2 \sup_{\ell \subset \mathbb{R}^d \text{ a line}} \int_\ell w$$

Conjecture 1.1 originally arose in the study of well-posedness for dispersive PDE ([Tak74; Tak80; Miz85]). Since then, the Mizohata-Takeuchi conjecture has taken on an important role in Fourier restriction theory for a few reasons, which we enumerate below.

1.1. **Multilinear Restriction Estimates.** In 2006, the following multilinear form of the restriction conjecture was formulated by [BCT06]:

Conjecture 1.3 (Multilinear Restriction). Let $\{U_j : j \in [d]\}$ be a collection of C^2 hypersurfaces in $\mathbb{R}^d$, so that the normal to U_j at any point is within $\frac{1}{100}$ of the x_j-axis. Let $\mathcal{E}_j$ denote the corresponding extension operators. Then, for each $\varepsilon > 0$, $q \geqslant \frac{2d}{d-1}$ and $p' \leqslant \frac{q(d-1)}{d}$,

$$\left\|\prod_{j=1}^{d} \mathcal{E}_j g_j\right\|_{L^{q/d}(B(0,R))} \lesssim \prod_{j=1}^{d} \|g_j\|_{L^p(U_j)}$$

for all $g_j \in L^p(U_j)$, $1 \leqslant j \leqslant d$, and all $R \geqslant 1$.

Currently, Conjecture 1.3 has been proven away from the endpoint ([Tao20]) and up to R^ε losses at the endpoint ([BCT06]).

In a recent paper, [CHV23] (see also [CHV18]) developed a general functional-analytic approach to dual formulations of multilinear estimates. In particular, they showed that The Mizohata-Takeuchi Conjecture implies Conjecture 1.3 without R^ε losses, using the endpoint multilinear Kakeya inequality of Guth ([Gut10]).

Theorem 1.2 therefore shows that it is not possible to use this approach to prove Conjecture 1.3.

1.2. Stein's Conjecture.
In 1978, it was suggested by Stein ([Ste79]) that Kakeya or Nikodym maximal functions may control the behavior of Bochner-Riesz multipliers (see [BCSV06] and the references therein for the history of this approach).

In the 1990's, several papers (see [CRS92; BRV97; CS97b; CS97a] and the references therein) were written on the subject, and the following conjecture has become known as Stein's conjecture:

Conjecture 1.4 (Stein's Conjecture). Under the hypotheses of Conjecture 1.1, the following holds:

$$\int_{\mathbb{R}^d} |\mathcal{E}f|^2 \, w \mathrm{d}x \lesssim \int_{\Sigma} |f(\varsigma)|^2 \sup_{\ell \| N(\varsigma)} Xw(\ell) \mathrm{d}\sigma(\varsigma) \tag{1.2}$$

It is worthwhile for historical purposes to note that Stein originally posed several different forms of this inequality, and only recently has Stein's conjecture (in the context of the extension operator) come to refer to the inequality above (see [BGNO24]).

The reason why one might expect (1.2) to hold is as follows. One often expects $\mathcal{E}f$ to be controlled in some sense by square functions, maybe of the form $\sum_{\Theta} |f|\hat{\Theta} \mathrm{d}\sigma|^2$ where Θ ranges over some collection of R^α-caps for some α. For weighted estimates on quantities of the form $\int_{\mathbb{R}^d} |\mathcal{E}^2(x)|w(x)\mathrm{d}x$, one expects the contribution from each term defining the square function to be concentrated along tubes by a parabolic rescaling argument. The norm $\|f\|_{L^2}^2$ measures the concentration of the L^2 mass of $\mathcal{E}f$ along these tubes in a way that is compatible with square functions, and so we are led to consider estimates of the form (1.2).

Note that Stein's conjecture would directly imply the Mizohata-Takeuchi conjecture since

$$\int_{\Sigma} |f(\varsigma)|^2 \sup_{\ell \| N(\varsigma)} Xw(\ell) \mathrm{d}\sigma(\varsigma) \leqslant \left(\int_{\Sigma} |f(\varsigma)|^2 \mathrm{d}\sigma(\varsigma)\right) \sup_{\varsigma \in \Sigma} \sup_{\ell \| N(\varsigma)} Xw(\ell) \mathrm{d}\sigma(\varsigma) = \|f\|_{L^2(\Sigma;\mathrm{d}\sigma)}^2 \|Xw\|_{L^\infty}$$

Thus, Theorem 1.2 implies that Stein's conjecture is false as stated in Conjecture 1.4.

1.3. The Kakeya Maximal Conjecture and the Restriction Conjecture.
The counterexample is also interesting because, assuming the Kakeya maximal conjecture, Stein's conjecture would imply the restriction conjecture ([BGNO24]).

Consider a Kakeya-type maximal operator $\mathcal{M}$ defined on suitably regular weights $w : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ by

$$\mathcal{M}w(\omega) = \sup_{T \| \omega} \int_T w, \qquad \omega \in \mathbb{S}^{n-1}$$

where T an infinitely long tube of width 1. We denote the local maximal function by $\mathcal{M}_R w = \mathcal{M}(w|_{B_R})$. Note that $\mathcal{M}_R$ is equivalent to the standard Kakeya maximal operator, upon a rescaling. As such, $\mathcal{M}_R$ is expected to satisfy the estimate

$$\|\mathcal{M}_R\|_{L^n(\mathbb{R}^n) \to L^n(\mathbb{S}^{n-1})} \lesssim_\varepsilon R^\varepsilon \tag{1.3}$$

which is equivalent to the Kakeya maximal conjecture.

To see how this is related to the restriction conjecture, take some $f \in L^{\frac{2n}{n-1}}(\Sigma)$ and note that

$$\|\mathcal{E}f\|^2_{L^{\frac{2n}{n-1}}} = \sup_{w \in L^n(\mathbb{R}^n)} \|w\|^{-1}_{L^n} \int_{\mathbb{R}^d} |\mathcal{E}f|^2(x) w(x) dx$$

by Hölder's inequality. If Stein's conjecture were to hold, we have

$$\|\mathcal{E}f\|^2_{L^{\frac{2n}{n-1}}} \lesssim \sup_{w \in L^n(\mathbb{R}^n)} \int_\Sigma |f(\xi)|^2 \mathcal{M}(N_\Sigma(\xi)) d\sigma(\xi)$$

From here, if (1.3) holds, then the above implies

$$\|\mathcal{E}f\|_{L^{\frac{2n}{n-1}}(B_R)} \lesssim_\varepsilon R^\varepsilon \|f\|_{L^{\frac{2n}{n-1}}(\Sigma)}$$

if Σ has nonvanishing curvature.

Note that the counterexample presented here only shows that Stein's conjecture is false up to a logarithmic factor; if Stein's conjecture were true locally, then the Kakeya maximal conjecture would still imply the restriction conjecture.

1.4. L^2-wellposedness of first-order pertubations of the Schrödinger equation.
The historical value of the Mizohata-Takeuchi conjecture stems from its applications to the well-posedness of first-order pertubations of the Schrödinger equation. Around 1980, Takeuchi ([Tak74],[Tak80]) was studying the Cauchy problem for operators of the form

$$L(u) = i\partial_t u + \triangle u + \sum_{j=1}^n b_j \partial_j u + c(x)u = f \tag{1.4}$$

The Cauchy problem for (1.4) is L^2-wellposed if, given any $u_0 \in L^2(\mathbb{R}^n)$ and $f \in C^0_t(L^2(\mathbb{R}^n))$, there is a unique solution $u(x,t)$ to (1.4) for $|t| < T$ that satisfies the estimate

$$\|u(\cdot,t)\|_{L^2} \lesssim_T \|u_0\|_{L^2} + \left| \int_0^t \|f(\cdot,s)\|_{L^2} \, ds \right| \tag{1.5}$$

Takeuchi ([Tak74; Tak80]) claimed that a sufficient condition for (1.5) to hold is the following:

$$\text{Re}\left(\int_0^t \sum_{j=1}^n b_j(x+s\nu)\nu_j ds \right) \lesssim 1$$

where the constant is independent of $(x,t,\nu) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{S}^{n-1}$. Later, Mizohata ([Miz85]) showed that Takeuchi's argument had an error.

It is well-known that the Cauchy problem for (1.4) is closely related to weighted L^2 estimates of the form (1.1).

1.5. **The intrinsic value of the Mizohata-Takeuchi Conjecture.** Lastly, it is worthwhile to note that there is some intrinsic value in the Mizohata-Takeuchi conjecture. As noted in [CIW24], much of the development in Fourier restriction theory has centered around the L^p-L^q mapping properties of $\mathcal{E}$, and the Mizohata-Takeuchi and Stein conjectures are some of the very few attempts to understand the shape of the level sets of $\mathcal{E}f$ for various $f : \Sigma \to \mathbb{C}$.

1.6. **Progress on the Mizohata-Takeuchi Conjecture.** In this section, we present a chronological overview of some important results in the direction of Conjecture 1.1. One of the first influential results was established by Barceló, Ruiz, and Vega ([BRV97]) which demonstrated the radial case of the Mizohata-Takeuchi conjecture (see also [CRS92]). This was achieved by directly estimating Bessel functions and the extension operator on spherical harmonics.

In 2009, Carbery ([Car09]) studied the nature of sets $U \subset \mathbb{R}^d$ where $X(1_U)$ is small. In general, the study of tube-occupancy of sets is difficult, independently of weighted Fourier extension estimates (see [DZ24]).

More recently, Shayya ([Sha23]) has studied the extent to which Mizohata-Takeuchi-type estimates can be derived from decay properties of $\widehat{\mathrm{d}\sigma}$ in the plane. Another important recent development was by Ortiz ([Ort23]), in which the Mizohata-Takeuchi conjecture was proven with an $R^{\frac{1}{4}+\varepsilon}$-loss for the cone in 3 dimensions.

Probably the most influential recent development in the direction of the Mizohata-Takeuchi conjecture is a result by Carbery, Iliopoulou, and Wang ([CIW24]), in which several special cases (for the weight) were proven, as well as the general conjecture with $R^{\frac{n-1}{n+1}+\varepsilon}$-loss. The approach was based on the refined decoupling estimates of Guth, Iosevich, Ou, and Wang ([GIOW20]).

The $R^{\frac{n-1}{n+1}}$-losses are especially relevant when placed in the context of a recent talk given by Guth ([Gut22]), in which he demonstrates that it is not possible to prove the Mizohata-Takeuchi conjecture without $R^{\frac{n-1}{n+1}}$ loss using only certain decoupling axioms.

More recently, Bennett, Gutierrez, Nakamura, and Oliveira ([BGNO24]) have explored connections to time-frequency analysis and have proven a related Mizohata-Takeuchi-type inequality involving Sobolev norms.

1.7. **Reformulations.** In light of Theorem 1.2, the following local version may be a plausible reformulation of the Mizohata-Takeuchi conjecture.

Conjecture 1.5 (Local Mizohata-Takeuchi). Under the same hypotheses as Conjecture 1.1, we have
$$\int_{B_R} |\mathcal{E}f(x)|^2 \, w(x)\mathrm{d}x \lesssim_{\varepsilon} R^{\varepsilon} \, \|f\|_{L^2(\Sigma;\mathrm{d}\sigma)}^2 \sup_{\ell \subset \mathbb{R}^d \text{ a line}} \int_{\ell} w$$

It is unclear whether one should expect Conjecture 1.5 to hold, or to expect an $R^{\frac{n-1}{n+1}}$-loss counterexample in the spirit of Guth's argument ([Gut22]).

1.8. **Outline of this paper.** We begin in Section 2 by proving L^2-based estimates on the X-Ray transform for positive measures. We then discuss how this can be used to formulate a weaker version of the Mizohata-Takeuchi conjecture on the Fourier side in Section 3. The construction relies on an incidence geometry Lemma, which is proven separately in Section 4.

1.9. **Acknowledgments.** The author would like to express her gratitude to her advisor, Ruixiang Zhang, for his generous introduction to the beautiful theory of Fourier restriction, as well as for pointing out the Mizohata-Takeuchi conjecture and reviewing earlier versions of this paper.

2. X-Ray transform estimates for positive measures

Let $\nu \in \mathbb{S}^{d-1}$. For any weight $w : \mathbb{R}^n \to [0, \infty)$, we define $X_\nu w(z) = \int_{\mathbb{R}\nu+z} w$ for any $z \in \nu^\perp$. For any function $\mu : \mathbb{R}^n \to \mathbb{C}$ so that μ restricts to an L^p function on every plane perpendicular to ν, we define the restriction map $P_\nu \mu : \mathbb{R} \to L^p(\nu^\perp)$ by $P_\nu \mu(\lambda)(\omega) = \mu(\lambda\nu + \omega)$.

In this section, we prove

Theorem 2.1. Let $h : \mathbb{R}^n \to \mathbb{C}$ be so that $|\hat{h}|^2 = w$, for a nonnegative weight $w : \mathbb{R}^n \to [0, \infty)$. Then we have the estimate

$$\|X_\nu w\|_{L^p} \leqslant \|P_\nu h\|_{L^2(L^q(\nu^\perp))}^2 \tag{2.1}$$

for $p \in [1, \infty]$ and $q = \frac{2p}{2p-1}$. In particular, when $p = \infty$ we have the estimate

$$\|X_\nu w\|_{L^\infty} \leqslant \|P_\nu h\|_{L^2(L^1(\nu^\perp))}^2$$

and in this case, the estimate is sharp when $h \geqslant 0$ everywhere.

Proof. The proof is relatively straightforward, once we digest the meaning of the norm appearing on the right of (2.1). We write

$$\|P_\nu h(\lambda)\|_{L^q(\nu^\perp)} = \left(\int_{\nu^\perp} |h(\lambda\nu + \omega)|^q \right)^{\frac{1}{q}}$$

Thus,

$$\|P_\nu h\|_{L^2(L^q(\nu^\perp))} = \left(\int_{\mathbb{R}} \left(\int_{\nu^\perp} |h(\lambda\nu + \omega)|^q \right)^{\frac{2}{q}} \right)^{\frac{1}{2}}$$

We note that $X_\nu w = \widehat{P_\nu \check{w}}(0)$ by the projection-slice theorem. We note that

$$P_\nu \check{w}(0) = P_\nu (h * \tilde{\hat{h}})(0) = \int_{\mathbb{R}} P_\nu h(\lambda) * \widetilde{P_\nu h(\lambda)}$$

where $\tilde{\mu}$ denotes $\mu \circ (x \mapsto -x)$ for any function μ. Therefore, we have

$$X_\nu w(z) = \int_{\mathbb{R}} \left| \widehat{P_\nu h(\lambda)}(z) \right|^2 \mathrm{d}\lambda$$

Minkowski's inequality gives

$$\|X_\nu w\|_{L^p(\nu^\perp)} \leqslant \int_{\mathbb{R}} \left\| \hat{P}_\nu h(\lambda) \right\|_{L^{2p}(\nu^\perp)}^2$$

Applying Hausdorff-Young yields (2.1). Note that, if $h \geqslant 0$ everywhere and $p = \infty$, then both Minkowski's inequality and Hausdorff-Young are sharp. $\square$

3. The proof of Theorem 1.2

This section is divided into four parts:

3.1. We discuss some preliminary estimates. This part is primarily devoted to posing similar problems of Mizohata-Takeuchi-type on the Fourier side.

3.2. We present a "white lie" proof of Theorem 1.2 to help the reader understand the important points.

3.3. We explain briefly an incidence lemma, which we prove in section 4.

3.4. We present a rigorous proof of Theorem 1.2.

3.1. **Some preliminaries.** Let $w : B_R \to [0, \infty)$ be a nonnegative weight. As [CIW24] notes, we are free to assume that w is locally constant at scale 1, and so w satisfies the necessary regularity conditions discussed in the previous section. We are interested in the quantity $E_w(f, g) : L^2(\Sigma; d\sigma) \times L^2(\Sigma; d\sigma) \to \mathbb{C}$ given below.

$$E_w(f, g) := \int_{\mathbb{R}^d} \mathcal{E}f(x)\overline{\mathcal{E}g}(x)w(x)dx = \left\langle w^{\frac{1}{2}}\mathcal{E}f d\sigma, w^{\frac{1}{2}}\mathcal{E}g d\sigma \right\rangle$$

We should think of E_w as a quadratic form on the Hilbert space $L^2(\Sigma; ds)$. The Mizohata-Takeuchi conjecture seeks an upper bound on the largest eigenvalue of E_w in terms of $\|Xw\|_{L^\infty}$.

As in the previous section, let $h : \mathbb{R}^n \to \mathbb{C}$ so that $\left|\hat{h}(x)\right|^2 = w(x)$. We write

$$E_w(f, g) = \langle h * f d\sigma, h * g d\sigma \rangle$$

In particular, we note $E_w(f, f) = \|h * f d\sigma\|_{L^2}^2$. Therefore, to prove Theorem 1.2, it suffices to prove the following.

Proposition 3.1. For any C^2 hypersurface $\Sigma \subset \mathbb{R}^d$, there exists for each $R \geqslant 1$, a function $f \in L^2(\Sigma; d\sigma)$ and a function $h : \mathbb{R}^d \to \mathbb{C}$ with $\hat{h} \subset B_R$, so that the following holds.

$$\|h * f d\sigma\|_{L^2}^2 \gtrsim \log R \|f\|_{L^2(\Sigma; d\sigma)}^2 \|P_\nu h\|_{L^2(L^1(\nu^\perp))}^2$$

for every $\nu \in \mathbb{S}^{d-1}$

In light of Proposition 3.1, we are led to consider partial progress towards Conjecture 1.5 in the following form.

Conjecture 3.2. For any C^2 hypersurface $\Sigma \subset \mathbb{R}^d$ with surface measure $d\sigma$ and any functions $f \in L^2(\Sigma; d\sigma)$ and $h : \mathbb{R}^d \to \mathbb{C}$ with $\hat{h} \subset B_R$, the following holds

$$\|h * f d\sigma\|_{L^2}^2 \lesssim_\varepsilon R^\varepsilon \|f\|_{L^2(\Sigma; d\sigma)}^2 \|P_\nu h\|_{L^2(L^1(\nu^\perp))}^2$$

for every $\nu \in \mathbb{S}^{d-1}$

To help the reader understand the construction of the counterexample, we present first a "white lie" version.

3.2. **A white lie construction.** Throughout this section, we use $A \approx B$ to mean that A and B are *essentially* equal, in the sense that B is a suitable approximation of A in some sense that we will formalize in a later section. The reader may feel free to assume $A = B$ when verifying estimates, even though $A = B$ is generally false. The point of this notation is to avoid excess detail at the expense of rigor.

Let $\xi_1, \cdots, \xi_N \subset \Sigma$ be a R^{-1}-separated collection of $N \sim \log R$ points in Σ. We will choose the ξ_i later on. Let us set $S_i = \left\|1_{B_{R^{-1}}(\xi_i) \cap \Sigma}\right\|_{L^1(\Sigma; d\sigma)}^{-1} 1_{B_R(\xi_i) \cap \Sigma} d\sigma$ and

$$f = \sum_{i=1}^N S_i.$$

We record the following for future use:

$$R^{-d+1}\|f\|_{L^2(\Sigma; d\sigma)}^2 \sim \|f\|_{L^1(\Sigma; d\sigma)} \sim N \tag{3.1}$$

The key step is the construction of the following lattice:

$$Q = \left\{ \vec{c} \cdot \vec{\xi} : \vec{c} \in \{0,1\}^N, c_1 + \cdots + c_N = \left\lfloor \frac{N}{2} \right\rfloor \right\}$$

Here, $\vec{c} \cdot \vec{\xi}$ denotes $c_1 \xi_1 + \cdots + c_N \xi_N$ by abuse of notation. We now define

$$h = R^d \sum_{q \in Q} 1_{B_{R^{-1}}(q)} \tag{3.2}$$

Note that $|Q| = \binom{N}{\lfloor \frac{N}{2} \rfloor} \sim N^{-\frac{1}{2}} 2^N$ by Stirling's approximation for the factorial.

The choice of scale R^{-1} reflects the locally constant principle:

White Lie 3.3. We have $\text{supp}(\hat{h}) \subset B(R)$.

We would like a lower bound on $\|f d\sigma * h\|_{L^2}^2$. The operation $h \mapsto f d\sigma * h$ sends each ball in (3.2) to the sum of the translations of that ball by each of $\xi_1, \cdots, \xi_N$. We will need an estimate of the following type.

White Lie 3.4. We have $S(\xi_i) * 1_{B(R^{-1})} \approx R^d 1_{B(\xi_i, R^{-1})}$

With this, we are free to write

$$h * f \approx \sum_{i=1}^N \sum_{q \in Q} R^d 1_{B(\xi_i + q, R^{-1})} \tag{3.3}$$

Now, note that $\geqslant \frac{1}{2}$ of the possible values of $\xi_i + q$ satisfy $c_i = 0$, if $q = \vec{c} \cdot \vec{\xi}$. Therefore, at least $\frac{1}{2}$ of the values of $\xi_i + q$ will lie in the lattice

$$Q' = \left\{ \vec{c} \cdot \vec{\xi} : \vec{c} \in \{0,1\}^N, c_1 + \cdots + c_N = \left\lfloor \frac{N}{2} \right\rfloor + 1 \right\},$$

which is a set of size at most $\binom{N}{\lfloor \frac{N}{2} \rfloor + 1} \sim |Q|$. Since there are a total of $N|Q|$ balls in (3.3), where each ball has an L^2 norm of $R^{\frac{d}{2}}$, we have

$$\|h * f\|_{L^2}^2 \gtrsim N^2 |Q| R^d \tag{3.4}$$

We would now like to estimate $\|P_\nu h\|_{L^2(L^1(\nu^\perp))}^2$. We recall

$$\|P_\nu h\|_{L^2(L^1(\nu^\perp))}^2 := \int_{\mathbb{R}} \|P_\nu h(\lambda)\|_{L^1}^2 \, d\lambda$$

Let $K_\nu(\lambda)$ denote the number of balls in the definition (3.2) of h that pass through the plane $\lambda \nu + \nu^\perp$, so that $\|P_\nu h(\lambda)\|_{L^1} \lesssim K_\nu(\lambda) R$. We deduce that $\|P_\nu h(\lambda)\|_{L^1}^2 \lesssim R^2 \|K_\nu\|_{L^2}^2$. We use $\|K_\nu\|_{L^1} \sim R^{-1} 2^N$ to obtain

$$\|P_\nu h(\lambda)\|_{L^1}^2 \lesssim \|K_\nu\|_{L^1} \|K_\nu\|_{L^\infty} \sim R 2^N \|K_\nu\|_{L^\infty} \tag{3.5}$$

Combining (3.1), (3.4), and (3.5), we obtain

$$\|h * f d\sigma\|_{L^2}^2 \|K_\nu\|_{L^\infty} \gtrsim N \|f\|_{L^2}^2 \|P_\nu h(\lambda)\|_{L^1}^2$$

We will show that $\|K_\nu\|_{L^\infty} \lesssim 1$ for every ν, for some choice of $\{\xi_1, \cdots, \xi_N\}$ in Lemma 3.5 below, i.e. no plane passes through more than ~ 1 balls in the definition (3.2) of h. The "white-lie" proof is now complete, contingent upon the following lemma.

3.3. An incidence geometry lemma.

Lemma 3.5. For any $R > 1$ sufficiently large, there exists

- A point $\xi_0 \in \Sigma$
- A set of $N \sim \log R$ points $\{\xi_1, \cdots, \xi_N\} \subset \Sigma$

so that the following conditions are met.

(i) No plane passes through more than 2^{d-1} balls in the collection of balls of radius R^{-1} around the points $c_1(\xi_1 - \xi_0) + \cdots + c_N(\xi_1 - \xi_0)$ for $\vec{c} \in \{0, 1\}^N$.

(ii) We have $|\xi_m - \xi_n| > R^{-1}$ for every $(m, n) \in [N]^2$.

We note that Q is a translate of $\{c_1(\xi_1 - \xi_0) + \cdots + c_N(\xi_1 - \xi_0) : \vec{c} \in \{0, 1\}^N, \sum_{i=1}^N c_i = \lfloor \frac{N}{2} \rfloor\}$ and so the point ξ_0 does not affect our estimates.

The proof of this Lemma is rather involved and may be of independent interest. We postpone the proof to Section 4.

3.4. A rigorous proof of Proposition 3.1.

We now present a rigorous proof of Proposition 3.1 by the means described in Section 3.2.

Let $\xi_1, \cdots, \xi_N$ with $N \sim \log R$ be a collection of points that satisfy the conclusion of Lemma 3.5. Let η be so that $\hat{\eta}$ is a smooth cutoff function with $1_{B(C^{-1})} \leq \hat{\eta} \leq 1_{B(C)}$ for some constant $C > 1$ to be determined later, so that $\hat{\eta}$ is supported in a ball of radius ~ 1; let $\eta_R(\xi) = R^d \eta(R\xi)$ be a rapidly decaying kernel at scale R^{-1}. For convenience, we assume that η is radially symmetric. We set

$$h_0 = \sum_{p \in Q} \delta_p,$$

and choose $h = h_0 * \eta_R$ so that $\hat{h} \subset B_{CR}$. Note that $\|h_0\|_{L^1} \sim \|h\|_{L^1} \sim |Q|$ and $\|h\|_{L^2}^2 \sim |Q|R^d$.

We set $f_i = \sum_i S_i$ as in the previous section. We would like to show the following estimate.

$$\|f \mathrm{d}\sigma * h\|_{L^2}^2 = \|f \mathrm{d}\sigma * h_0 * \eta_R\|_{L^2}^2 \gtrsim N^2 |Q| R^d \tag{3.6}$$

We have

$$\|f * h_0 * \eta_R\|_{L^2}^2 = \int_{\Sigma \times \Sigma} \sum_{i,j=1}^N \sum_{q,q' \in Q} S_i(\varsigma) S_j(\varsigma') [\eta_R * \eta_R](\varsigma - \varsigma' + q - q') \mathrm{d}\sigma(\varsigma) \mathrm{d}\sigma(\varsigma') \tag{3.7}$$

Note that $\eta_R * \eta_R = \widetilde{(\widehat{\eta_R})^2}$ is another rapidly decaying kernel at scale R^{-1}. We morally have $\eta_R * \eta_R \lesssim 1_{B_{2R^{-1}}}$ but the honest kernel has rapidly decaying oscillatory tails. If we set

$$H = \sum_{i,j=1}^N \sum_{q,q' \in Q} \delta_{\xi_i - \xi_j + q - q'} * T_{ij}$$

where $T_{ij}(\xi_j - \xi_i + \zeta) = S_i * \tilde{S}_j(\zeta)$, then we can rewrite (3.7) as

$$\|f * h_0 * \eta_R\|_{L^2}^2 = \langle H, \eta_R * \eta_R \rangle,$$

Note that if q, ξ_i, and ξ_j are randomly chosen, then there is a $\geq \frac{1}{4}$ probability that $q + \xi_i - \xi_j$ will lie in Q. Therefore, we deduce

$$\int_{B_{R^{-1}}(0)} H \gtrsim N^2 |Q| \tag{3.8}$$

Now, note that we can deduce that that points in Q are separated by at least R^{-1} from Lemma 3.5, in the sense that every R^{-1}-ball contains $\lesssim 1$ of the points (in fact, their ν-projections

are separated). Since $\|T_{ij}\|_{L^1} \sim 1$, T_{ij} is nonnegative, and T_{ij} is supported in $B_{2R^{-1}}(0)$, we deduce that

$$\int_{B_{R^{-1}}(\zeta)} H \lesssim N^2 |Q|. \tag{3.9}$$

for any $\zeta \in \mathbb{R}^d$.

Combining (3.8) and (3.9), we conclude that (3.6) holds for a suitable choice of η so that $\eta * \eta \geqslant B_1(0)$ and $\int_{\mathbb{R}^d} (\eta * \eta)_- < c$ for some sufficiently small c (here, $\eta * \eta = (\eta * \eta)_+ - (\eta * \eta)_-$ is a difference of nonnegative functions).

To estimate $\|P_\nu h\|^2_{L^2(L^1(\nu^\perp))}$, we note that the method from Section 3.2 must be modified slightly, since η has a rapidly decaying tail, and therefore we expect some interference between the different "layers" of Q. More concretely, we write

$$\|P_\nu h(\lambda)\|_{L^1} \lesssim \int_{\mathbb{R}} K_\nu(\lambda') \mu_R(\lambda - \lambda') d\lambda' = K_\nu * \mu_R(\lambda)$$

where K_ν is defined as in Section 3.2 and $\mu_R(\lambda) = R\mu(R\lambda)$ is some rapidly decaying smooth kernel at scale R^{-1}. We write

$$\begin{aligned}
\|P_\nu h\|^2_{L^2(L^1(\nu^\perp))} &\lesssim \|RK_\nu * \mu_R\|^2_{L^2} \\
&\leqslant R^2 \|K_\nu\|_{L^1} \|K_\nu\|_{L^\infty} \|\mu_R\|^2_{L^1} \\
&\lesssim R^2 \|K_\nu\|_{L^1} \|K_\nu\|_{L^\infty} \\
&\lesssim R|Q|
\end{aligned}$$

This completes the proof.

4. The proof of Lemma 3.5

Let $\Sigma \subset \mathbb{R}^d$ be a compact C^2 hypersurface that is not a subset of a plane. We will see in section 4.4 that Lemma 3.5 will follow easily from the following Lemma.

Lemma 4.1. For any $R > 1$ sufficiently large, there exists

- A point $\xi_0 \in \Sigma$
- A set of $N \sim \log R$ points $\{\xi_1, \cdots, \xi_N\} \subset \Sigma$

so that the following conditions are met.

(i) For any direction $\nu \in \mathbb{S}^{d-1}$, the ν-projections of the differences $\xi_m - \xi_1$ lie in nearly distinct dyadic intervals; that is, if $\mathfrak{u}_n$ denotes the part of the partition $\mathfrak{U}^1(1)$ that contains $\log_2(|\pi_\nu(\xi_n - \xi_0)|)$, then we have

$$\mathfrak{u}_m = \mathfrak{u}_n \text{ and } m > n \implies n \in S_\nu \tag{4.1}$$

where S_ν is called the set of bad values of n. The size of S is at most $d - 1$.

(ii) We have $|\xi_m - \xi_n| > R^{-1}$ for every $(m, n) \in \{1, N\}^2$.

Remark 4.2. It is much easier to see that the lemma is true if we are allowed to choose different ξ_n for different ν; since Σ does not lie in a hyperplane, it cannot project to a point, so $\pi_\nu(\Sigma)$ must contain some line segment from which we can choose ξ_n dyadically.

Before proving Lemma 4.1, we explain the main ideas. First, the condition that Σ is a hypersurface is not necessary as long as we assume that it does not project to a point and is C^2. We will also see that higher order corrections do not play an important role in the proof, so we might as

well assume that Σ is smooth. In this case, the desirable region of Σ is best approximated by the *moment curve* $\mathcal{M}_d$.

$$\mathcal{M}_d(t) = (t, t^2, \cdots, t^d) \tag{4.2}$$

That is, we should be able to construct $\xi_n \approx \mathcal{M}_d(t_n)$ for a nice choice of t_n and most of the $\mathcal{M}_d(t_n)$ will nearly lie in Σ. Note that if $\nu = (\nu_1, \cdots, \nu_d)$ was close to $(1, \cdots, 0)$, say $|\nu_1| > c > 0$, then the projection $\pi_\nu(\mathcal{M}_d(t))$ would look like a line near $t = 0$. Motivated by this, we might decide to set $t_n = 2^{-n}$ dyadically. This also works well if, e.g., $\nu = (0, 1, \cdots, 0)$ because then $\pi_\nu(\mathcal{M}_d(t_n)) \sim 2^{-2n}$. It is remarkably harder to prove that the estimates are uniform over all possible choice of lines.

It is, in general, difficult to work with neighborhoods of the projections $\pi_\nu(\mathcal{M}_d(t_n))$ and it is usually preferable to work with neighborhoods of t_n prior to projection. We might try setting $B_n = B(\mathcal{M}_d(t_n), 2|\mathcal{M}_d(t_{n-1})|)$ so that $0 \notin \pi_\nu(B_n)$ implies that $\pi_\nu(\mathcal{M}_d(t_n))$ lies in a dyadic interval higher than $\mathcal{M}_d(|t_{n-1}|)$, and therefore higher than $\pi_\nu(\mathcal{M}_d(t_{n-1}))$. The advantage here is that the problem reduces to showing something about how many balls lie on a hyperplane $\nu^\perp$ through the origin, which seems much more tractable.

Unfortunately, this approach fails. To see why, let us choose $t_n = c^{-n}$ for some $c > 1$, and set $d = 2$ for definiteness, so that the B_n are balls of radius $\sim c^{-1}|t_n| = c^{-1-n}$. If we choose $\nu = (0, 1)$, then the B_n project to segments of length $\sim c^{-1-n}$ a distance $\sim c^{-2n}$ away from 0. As n gets large, these segments will overlap a lot because the lengths do not decay fast enough.

In light of the above setback, we might be temped to choose the B_n to be rectangles instead. If we set $U_n = t_n + [-2c^{-1-n}, 2c^{-1-n}] \times [-2c^{-2-2n}, 2c^{-2-2n}]$, then we might hope that $\pi_\nu(U_n)$ contains the origin when $\pi_\nu(t_n)$ does not lie in a dyadic interval higher than $\pi_\nu(t_{n-k})$. That is, if $\frac{\pi_\nu(t_n)}{\pi_\nu(t_{n-k})} \in [-2, 2]$ for some $k > 1$, we would like to see if $0 \in \pi_\nu(U_n)$. If $\pi_\nu(t_n) = \alpha \pi_\nu(t_{n-k})$ for some $\alpha \in [-2, 2]$, then we would like to show that $t_n - \alpha t_{n-k} \in U_n$ because we know that $\pi_\nu(t_n - \alpha t_{n-k}) = 0$. As it turns out, it is not hard to verify that $t_n - \alpha t_{n-k} \in U_n$ from the definition of U_n.

When $d > 2$, the main idea of the proof is to use a projection $\varphi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ to induct on d. We define φ below.

$$\varphi(x_1, \cdots, x_d) = \frac{(x_2, \cdots, x_d)}{x_1} \tag{4.3}$$

We will see that φ is useful because it reduces the dimension by 1 and it sends hyperplanes in $\mathbb{R}^d$ to (affine) hyperplanes in $\mathbb{R}^{d-1}$. This is the point where the choice (4.2) of the moment curve is essential because $\varphi \circ \mathcal{M}_d = \mathcal{M}_{d-1}$. After suitably defining the rectangles U_n^d in dimension $d > 2$, we will note that $0 \in \pi_\nu(U_n^d)$ is equivalent to the statement that the hyperplane $\nu^\perp$ that contains 0 also passes through some point of U_n^d. This will then imply that $\varphi(\nu^\perp)$ meets some point of $\varphi(U_n^d)$. However, $\varphi(\nu^\perp)$ is just a hyperplane in $\mathbb{R}^{d-1}$ and we will see that $\varphi(U_n^d)$ is very close to $U_{n,d-1}$ and so the problem reduces to a $d-1$-dimensional problem.

There is one more complication which arises, namely that after applying φ, we are dealing with hyperplanes in general position. Before applying φ, we were only dealing with hyperplanes through the origin.[1] To remedy this, consider fixing a scale $\mathsf{d}_n \sim |t_n|$ for some n. Note that either $\varphi(\nu^\perp)$ will be very far away from the origin, it will be very close to the origin, or it will be a moderate distance away from the origin. If $\varphi(\nu^\perp)$ is very far from the origin at scale d_n, then there is no risk of $\varphi(\nu^\perp)$ passing through $\varphi(U_n^d)$. On the other hand, if $\varphi(\nu^\perp)$ is very close to the origin at scale d_n,

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[1]It is notable that this is the only reason that the number of "bad" n, i.e. the constant in (4.1), is dependent on d.

then we might as well assume that it meets the origin, in which case we have reduced the problem completely to the $n-1$-dimensional case. For the third case, namely that the distance from $\varphi(\nu^\perp)$ to the origin is the same scale as d_n, we are truly without hope. Luckily, it is not possible for very many t_n to meet this fate since there are not many scales d_n that are close to the distant from $\varphi(\nu^\perp)$ to the origin.

In view of the principles above, we are ready to formalize the proof.

4.3. **The proof of Lemma 4.1.** We divide the proof into two lemmas. Lemma 4.3 is the main argument for the moment curve and Lemma 4.4 proves that Lemma 4.1 for an arbitrary curved C^2 hypersurface Σ reduces to the moment-curve case.

Recall the definition of the moment curve.

$$\mathcal{M}_d(t) = (t, t^2, \cdots, t^d) \tag{4.2}$$

For any $c > 1$, we set $t_{c,n} = c^{-n}$ and $x_{c,n} = \mathcal{M}_d(t_{c,n})$. Throughout the proof, we will use the rescaling symmetry of the moment curve.

$$\mathcal{M}_d(c^{-1}t) = L_c \circ \mathcal{M}_d(t)$$

Here, $L_c(x)$ is the entry-wise multiplication of x by $\mathcal{M}_d(c^{-1})$. Notably, we have

$$L_c(x_{c,n}) \;=\; L_c(x_{c,n+1})$$

We also recall the definition of $\varphi : \mathbb{R}^d$.

$$\varphi(x_1, \cdots, x_d) = \frac{(x_2, \cdots, x_d)}{x_1} \tag{4.3}$$

At the core of the argument is the fact that φ and L_c commute:

$$\varphi \circ L_c = L_c \circ \varphi$$

Here, we written L_c for the corresponding scaling in $\mathbb{R}^{d-1}$ by abuse of notation. In fact, by a similar abuse of notation, all of L_c, $\mathcal{M}$, and φ mutually commute.

Given any vector v, let us denote by $Q(v)$ the axially-oriented rectangular box centered at the origin, whose dimensions are encoded in the entries of the vector $2v$. We should imagine v as a corner of the box. We now define the boxes $U^d_{b,c,n}$, where $b > 1$ will be fixed later.

$$U^d_{b,c,n} = x_{c,n} + Q(\mathcal{M}_d(bt_{c,n+1})) \tag{4.4}$$

In the notes following Remark 4.2, we took $b = 2$. To digest (4.4), recall that we are looking for the smallest box so that translations $x_{c,n} + \alpha x_{c,k}$ remain in the box, for suitably defined α.

Since Q commutes with axial rescaling, the boxes $U^d_{b,c,n}$ are related by L_c.

$$L_c(U^d_{b,c,n}) \;=\; U^d_{b,c,n+1} \tag{4.5}$$

We are now ready to state and prove the following lemma.

Lemma 4.3. Let d be arbitrary. For $c \gg b \gg 1$ sufficiently large, we have the following.

(i) For any direction $\nu \in \mathbb{S}^{d-1}$ and any $(n, k) \in \mathbb{Z}^2$ with $n < k$, the projections $\pi_\nu\left(U^d_{b,c,n}\right)$ and $\pi_\nu\left(U^d_{b,c,k}\right)$ do not overlap unless $n \in S_\nu$. Here, S_ν is called the set of bad n and $|S_\nu| \leqslant d-1$.

(ii) In particular, the values of $\log_c \pi_\nu\left(c_{c,n}\right)$ are distinct, except for a set of as most $d-1$ choices of n.

Proof. The proof is by induction on d. Note that the $d = 1$ case is trivial.

The first step is to note that $\pi_\nu\left(U_{b,c,n}^d\right)$ and $\pi_\nu\left(U_{b,c,k}^d\right)$ overlap iff $\nu^\perp$ passes through a point of $U_{b,c,n}^d - U_{b,c,k}^d$. Note that all of the boxes $Q(\mathcal{M}_d(bt_{c,k+1}))$ in the definition (4.4) of $U_{b,c,n}^d$ will fit inside $Q(\mathcal{M}_d(bt_{c,n+1}))$. The points $x_{c,k}$ also fit inside this box, so we deduce the following.

$$U_{b,c,n}^d - U_{b,c,k}^d \subset U_{b_1,c,n}^d$$

Here, $b_1 \sim b$ is some other scale. Therefore, it suffices to construct a set S_ν of bad points so that for $n \notin S_\nu$, the hyperplane $\nu^\perp$ will not meet $U_{b_1,c,n}^d$. This is equivalent to proving that $\varphi(\nu^\perp)$ does not meet $\varphi(U_{b_1,c,n}^d)$ because $\varphi(U_{b_1,c,n}^d)$ is well-defined if $c > b_1$.

We observe that (4.5) in dimension $d - 1$ implies the following:

$$\varphi(U_{b_1,c,n}^d) = L_c^n(\varphi(U_{b_1,c,1}^d)) \tag{4.6}$$

Note that $\varphi(U_{b_1,c,1}^d)$ fits inside a box $U_{b_2,c,1}^{d-1}$ for some $b_2 \sim b_1$ now that we are at scale $b_1 \times \cdots \times b_1$. By (4.5) in dimension d with (4.6), we deduce that the box construction of U nearly commutes with φ. That is,

$$\varphi(U_{b_1,c,n}^d) \subset U_{b_2,c,n}^{d-1}$$

Therefore, it suffices to show that $\varphi(\nu^\perp)$ does not meet more than $d - 1$ of the boxes $U_{b_2,c,n}^{d-1}$. We would be done by the induction hypothesis if $\varphi(\nu^\perp)$ were to pass through the origin, so we assume that $0 \notin \varphi(\nu^\perp)$. Consider boxes $Q(\mathcal{M}_{d-1}(b_3 t_{c,m+1}))$ and take m_0 to be maximal so that $\varphi(\nu^\perp)$ contains a point l inside such a box. Here, $b_3 \sim b_2$ is so that the boxes $U_{b_2,c,k}^{d-1}$ lie in $Q(\mathcal{M}_{d-1}(b_3 t_{c,m+1}))$ for $k > m + 1$. We observe that, for $n > m_0 + 1$, the hyperplane $\varphi(\nu^\perp)$ does not meet $U_{b_2,c,n}^{d-1}$ by maximality of m_0.

If $n = m_0 + 1$, then it does not matter if $\varphi(\nu^\perp)$ meets $U_{b_2,c,n}^{d-1}$ if we can show that at most $d - 2$ of the values $n < m_0$ are bad.

If $n < m_0 + 1$, we observe that $\varphi(\nu^\perp) = l + \nu_2^\perp$, where ν_2 is perpendicular to $\varphi(\nu^\perp)$. Thus, it suffices to show that no more than $d - 2$ of the translated boxes $U_{b_2,c,n}^{d-1} - l$ meet $\nu_2^\perp$. However, since $l \in Q(\mathcal{M}_{d-1}(b_3 t_{c,m+1}))$, we deduce that $U_{b_2,c,n}^{d-1} - l \subset U_{b_4,c,n}^{d-1}$ for some $b_4 \sim b_2$. Our inductive hypothesis then implies that at most $d - 2$ of the boxes $U_{b_4,c,n}^{d-1}$ meet $\nu_2^\perp$. This completes the proof. $\qquad\square$

We are now ready to handle the general case of a curved C^2 hypersurface Σ. We assume that Σ is given near $\xi = 0$ as the graph of a C^2 function $\Phi : \mathbb{R}^{d-1} \to \mathbb{R}$. We write

$$\Phi(\omega) = C(\omega, \omega) + o(|\omega^2|) \tag{4.7}$$

where C is some quadratic form. We assume additionally that the largest-magnitude eigenvalue of C corresponds to an eigenvector pointing in the $\eta_1 = (1, 0, \cdots, 0)$ direction. It is also convenient to assume by rescaling that $C(\eta_1, \eta_1) = 1$. We will now set

$$\omega_n = \left(c^{-n}, c^{-3n}, c^{-4n}, \cdots, c^{-dn}\right)$$

for some $c > 1$. We set $\xi_n = \Phi(\omega_n)$.

We are now ready to prove Lemma 4.1 for Σ.

Lemma 4.4. For sufficiently large $n > 0$, each ξ_n as defined above lies in the box $U_{b,c,n}^d$ for some choice of b, c consistent with Lemma 4.3.

Proof. Suppose we set $\tilde{\mathcal{M}}_d(t) = \Phi(t, t^3, t^4, \cdots, t^d)$ so that $\xi_n = \tilde{\mathcal{M}}_d(c^{-n})$. Note that $\tilde{\mathcal{M}}_d(t) - \mathcal{M}_d(t)$ lies entirely in the $(0, 1, \cdots, 0)$-direction, so it suffices to prove that this difference lies within the $(0, 1, \cdots, 0)$-width of $U_{b,c,n}^d$. Recall that this width is $O(|t^2|)$, so it suffices to show that

$$\tilde{\mathcal{M}}_d(t) - \mathcal{M}_d(t) = o(|t^2|)$$

However, we have

$$B((t, 0, \cdots, 0), (t, 0, \cdots, 0)) = t^2$$

and

$$|(0, t^3, t^4, \cdots, t^d)| = O(|t|^3)$$

Since the derivative of $\omega \to B(\omega, \omega)$ tends to zero as $\omega \to 0$, we conclude that

$$B((t, t^3, t^4, \cdots, t^d), (t, t^3, t^4, \cdots, t^d)) = B((t, 0, \cdots, 0), (t, 0, \cdots, 0)) + o(|t|^3) = t^2 + o(|t|^2)$$

by writing $(t, t^3, t^4, \cdots, t^d) = (t, 0, \cdots, 0) + (0, t^3, t^4, \cdots, t^d)$.

This completes the proof upon applying (4.7). $\qquad\square$

To finish the proof of Lemma 4.1, note that we can relabel the ξ_n to assume that n is sufficiently large. The pigeonholing condition (4.1) is then satisfied by choosing c sufficiently large. The only step left is then proving that $R^{-1} \sim R^{-1}$. The smallest separation between two values of ξ_n will occur when n reaches its maximum, i.e. $n \sim N$. In particular, the minimum separation is $\gtrsim |t_N| \sim c^{-N}$. We can ensure that $c^{-N} \sim R^{-1}$ by choosing the right constant in $N \sim \log R$.

Remark 4.5. There is actually another proof of Lemma 4.1 that relies on a polynomial-partitioning technique. Instead of inducting on the dimension d, one can project the moment curve onto a line to obtain a polynomial whose coefficients are suitably bounded on both sides. The problem then reduces to a problem about the shape of sets where the polynomial and its derivatives have certain signs. However, this polynomial-based approach requires some delicate estimates on the shapes of these sets and how these shapes interact with the corresponding shapes for its derivative. It is also very difficult to yield the sharp estimate of $d-1$ for the number of bad points using this approach, and it is unclear how to construct the sharp examples for ν. It is for these reasons that the approach included is preferred.

4.4. **The proof of Lemma 3.5 using Lemma 4.1.** To prove the plane condition Lemma 3.5 (i), it suffices to show that the projections $\pi_\nu \left(\vec{c} \cdot \vec{\xi} \right)$ are distinct at scale R^{-1}. Note that, even though Lemma 4.1 gave only distinct values of $\log_2 \xi_i$, we can easily guarantee $\log_K \xi_i$ for any $K > 0$ using only every $\log_2(K)$th value of ξ_i. Let us fix $\vec{c} \in \{0, 1\}^N$. If

$$\left| \pi_\nu \left(\vec{c} \cdot \vec{\xi} \right) - \pi_\nu \left(\vec{c}' \cdot \vec{\xi} \right) \right| \leqslant R^{-1} \tag{4.8}$$

holds for more than 2^{d-1} values of $\vec{c}$, then (4.8) holds for at least one $\vec{c}'$ so that $c_m' \neq c_m$ for some $m \notin S_\nu$, that is, m is not a bad value, and $c_n' = c_n$ for all $n \in S_\nu$. Let us assume that m is the value in $\{1, \cdots, N\}$ where $\pi_\nu (\xi_m - \xi_0)$ is maximized. Next, note that since the values of $\pi_\nu (\xi_k - 0)$ lie in distinct logarithmic intervals of the form $[\alpha, \alpha K]$, we deduce that $|\pi_\nu (\xi_m - \xi_0)|$ is more than twice the sum of all $|\pi_\nu (\xi_k - \xi_0)|$ for k with $c_k \neq c_k'$. This contradicts (4.8) by the triangle inequality applied to $\sum_l \pm \pi_\nu (\xi_l - \xi_0)$, where the sum is over all values of l where $c_l \neq c_l'$.

References

[BBC08]   Juan Antonio Barceló, Jonathan Bennett, and Anthony Carbery. "A note on localised weighted inequalities for the extension operator". In: *Journal of the Australian Mathematical Society* 84.3 (2008), pp. 289–299.

[BCSV06] Jonathan Bennett, Anthony Carbery, Fernando Soria, and Ana Vargas. "A Stein conjecture for the circle". In: *Mathematische Annalen* 336 (2006), pp. 671–695.

[BCT06]   Jonathan Bennett, Anthony Carbery, and Terence Tao. "On the multilinear restriction and Kakeya conjectures". In: *Acta Mathematica* 196.2 (Jan. 2006), pp. 261–302. DOI: `10.1007/s11511-006-0006-4`.

[BGNO24] Jonathan Bennett, Susana Gutierrez, Shohei Nakamura, and Itamar Oliveira. "A phase-space approach to weighted Fourier extension inequalities". In: *arXiv preprint arXiv:2406.14886* (2024). URL: `https://arxiv.org/pdf/2406.14886`.

[BN21]   Jonathan Bennett and Shohei Nakamura. "Tomography bounds for the Fourier extension operator and applications". In: *Mathematische Annalen* 380 (2021), pp. 119–159. URL: `https://arxiv.org/pdf/2001.01674`.

[BRV97]   Juan Antonio Barceló, Alberto Ruiz, and Luis Vega. "Weighted estimates for the Helmholtz equation and some applications". In: *journal of functional analysis* 150.2 (1997), pp. 356–382.

[Car09]   Anthony Carbery. "Large sets with limited tube occupancy". In: *Journal of the London Mathematical Society* 79.2 (2009), pp. 529–543. DOI: `https://doi.org/10.1112/jlms/jdn086`. eprint: `https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/jlms/jdn086`. URL: `https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/jlms/jdn086`.

[CHV18]   Anthony Carbery, Timo S Hänninen, and Stefán Ingi Valdimarsson. "Multilinear Duality and Factorisation for Brascamp-Lieb-type Inequalities with applications". In: *arXiv preprint arXiv:1809.02449* (2018).

[CHV23]   Anthony Carbery, Timo S Hänninen, and Stefán Ingi Valdimarsson. "Disentanglement, multilinear duality and factorisation for nonpositive operators". In: *Analysis & PDE* 16.2 (2023), pp. 511–543. URL: `https://arxiv.org/abs/2003.03326v2`.

[CIW24]   Anthony Carbery, Marina Iliopoulou, and Hong Wang. "Some sharp inequalities of Mizohata–Takeuchi-type." In: *Revista Mathematica Iberoamericana* 40.4 (2024). URL: `https://arxiv.org/pdf/2302.11877`.

[Cor77]   Antonio Cordoba. "The Kakeya maximal function and the spherical summation multipliers". In: *American Journal of Mathematics* 99.1 (1977), pp. 1–22.

[CRS92]   Anthony Carbery, Elena Romera, and Fernando Soria. "Radial weights and mixed norm inequalities for the disc multiplier". In: *Journal of functional analysis* 109.1 (1992), pp. 52–75.

[CS97a]   Anthony Carbery and Fernando Soria. "Pointwise Fourier inversion and localisation in R n". In: *Journal of Fourier Analysis and Applications* 3 (1997), pp. 847–858.

[CS97b]   Anthony Carbery and Fernando Soria. "Sets of divergence for the localization problem for Fourier integrals". In: *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics* 325.12 (1997), pp. 1283–1286.

[CSV07]   Anthony Carbery, Fernando Soria, and Ana Vargas. "Localisation and weighted inequalities for spherical Fourier means". In: *Journal d'Analyse Mathématique* 103 (2007), pp. 133–156.

[DZ24]   Ciprian Demeter and Ruixiang Zhang. "On the N-set occupancy problem". In: *arXiv preprint arXiv:2403.10678* (2024).

[GIOW20]   Larry Guth, Alex Iosevich, Yumeng Ou, and Hong Wang. "On Falconer's distance set problem in the plane". In: *Inventiones mathematicae* 219.3 (2020), pp. 779–830.

[Gut10]   Larry Guth. "The endpoint case of the Bennett–Carbery–Tao multilinear Kakeya conjecture". In: *Acta Mathematica* 205.2 (Jan. 2010), pp. 263–286. DOI: `10.1007/s11511-010-0055-6`. URL: `https://arxiv.org/abs/0811.2251`.

[Gut22]   Larry Guth. "An enemy scenario in restriction theory". In: *joint talk for AIM Research Com- munity Fourier restriction conjecture and related problems and HAPPY network* (2022). URL: `https://www.youtube.com/watch?v=x-DET83UjFg`.

[Miz85]   Sigeru Mizohata. *On the Cauchy problem*. Vol. 3. Academic Press, 1985.

[Ort23]   Alexander Ortiz. "A sharp weighted Fourier extension estimate for the cone in $\mathbb{R}^3$ based on circle tangencies". In: *arXiv preprint arXiv:2307.11731* (2023). URL: `https://arxiv.org/pdf/2307.117`

[Sha23]   Bassam Shayya. "Mizohata–Takeuchi estimates in the plane". In: *Bulletin of the London Mathematical Society* 55.5 (2023), pp. 2176–2194. URL: `https://arxiv.org/pdf/2208.10305`.

[Ste79]   Elias M Stein. "Some problems in harmonic analysis". In: *Harmonic analysis in Euclidean spaces, Proceedings of the Symposium in Pure Mathematics of the Amer. Math. Soc., Williams College, Mass, Proc. Sympos. Pure Math., XXXV Part I, 1979.* Amer. Math. Soc. 1979, pp. 3–20.

[Tak74]   Jiro Takeuchi. "A necessary condition for the well-posedness of the Cauchy problem for a certain class of evolution equations". In: *Proceedings of the Japan Academy* 50.2 (1974), pp. 133–137.

[Tak80]   Jiro Takeuchi. "On the Cauchy problem for some non-kowalewskian equations with distinct characteristic roots". In: *Journal of Mathematics of Kyoto University* 20.1 (1980), pp. 105–124.

[Tao20]   Terence Tao. "Sharp bounds for multilinear curved Kakeya, restriction and oscillatory integral estimates away from the endpoint". In: *Mathematika* 66.2 (2020), pp. 517–576. URL: `https://arxiv.org/abs/1907.11342`.

[WZ25]   Hong Wang and Joshua Zahl. "Volume estimates for unions of convex sets, and the Kakeya set conjecture in three dimensions". In: *arXiv preprint arXiv:2502.17655* (2025).